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The Existence of a Positive Solution for a Nonlinear Fractional Differential Equation

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We prove existence and uniqueness theorems for a nonlinear fractional differential equation.

1. INTRODUCTION

In this paper we shall consider the nonlinear fractional equation

$$D^s u = f(t, u), \qquad 0 < t < 1, \tag{1}$$

where $0 < s < 1$, D^s is the standard Riemann–Liouville fractional derivative, and f: $[0,1] \times [0, +\infty) \to [0, +\infty)$ is a given continuous function.

Many papers and books on fractional calculus differential equations have appeared recently. Most of them are devoted to the solvability of the linear fractional equation in terms of a special function (see, for example, Miller and Ross [1] and Campos [2]) and to problems of analyticity in the complex domain (see, for example, Ling and Ding [3]). Moreover, some have been devoted to the solvability of nonlinear fractional differential equations since 1996 specifically (see D. Delbosco and L. Rodino [4]). No contribution exists, as far as we know, concerning the existence of a positive solution for nonlinear fractional equations of the form

$$D^s u = f(t, u), \qquad 0 < s < 1, 0 < t < 1.$$

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In [4], D. Delbosco and L. Rodino considered the existence of a solution for the nonlinear fractional differential equation $D^s u = f(t, u)$, where $0 < s < 1$, and $f\colon [0, a] \times R \to R$, $0 < a \leq +\infty$ is a given function, continuous in $(0, a) \times R$. They obtained results for solutions by using the Schauder fixed theorem and the Banach contraction principle. Here, we consider the existence of a positive solution for Eq. (1), by using the fixed theorem for the cone that the paper on fractional differential equations didn't take frequently.

Concerning the definitions of a fractional integral and derivative and related basic properties used in the text see [4, Sect. 2].

This paper is organized as follows. In Section 2 we consider the existence of positive solution for Eq. (1) by using the sub- and super-solution method. Section 3 contains results for uniqueness of a positive solution.

Let $X = C[0,1]$ be the Banach space endowed with the sup norm and define the cone

$$K = \{u \in X;\ u(t) \geq 0,\ 0 \leq t \leq 1\}.$$

The positive solution which we consider in this paper is such that $u(0) = 0$, $u(t) > 0$, $0 < t \leq 1$, $u \in X$.

2. RESULT

In this section, we consider the existence of a positive solution for Eq. (1).

According to [4, Proposition 2.4], Eq. (1) is equivalent to the integral equation

$$u(t) = I^s f(t, u(t)) = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, u(\tau))\, d\tau, \qquad (2.1)$$

where Γ denotes the gamma function.

Let $T\colon K \to K$ be the operator defined as

$$Tu(t) = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, u(\tau))\, d\tau.$$

We have the following lemma.

LEMMA 2.1. *The operator $T\colon K \to K$ is completely continuous.*

Proof. The operator $T\colon K \to K$ is continuous in view of the assumption of nonnegativeness and continuity of f.

Let $M \subset K$ be bounded; i.e., there exists a positive constant l such that $\|u\| \le l \ \forall \ u \in M$.

Let $L = \max_{0 \le t \le 1,\, 0 \le u \le l} f(t, u(t)) + 1$. Then $\forall \ u \in M$, we have

$$\begin{aligned} |Tu(t)| &\le \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} |f(\tau, u(\tau))| \, d\tau \\ &\le \frac{L}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} \, d\tau \\ &= \frac{L}{\Gamma(1+s)} t^s \\ \|Tu\| &\le \frac{L}{\Gamma(1+s)}. \end{aligned}$$

Hence $T(M)$ is bounded.

For each $u \in M$, $\forall \ \varepsilon > 0$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, let $\delta = (\varepsilon \Gamma(1+s)/2L)^{1/s}$. Then when $t_2 - t_1 < \delta$, we have

$$\begin{aligned} &\left| Tu(t_1) - Tu(t_2) \right| \\ &\quad = \left| \frac{1}{\Gamma(s)} \int_0^{t_1} (t_1 - \tau)^{s-1} f(\tau, u(\tau)) \, d\tau \right. \\ &\qquad \left. - \frac{1}{\Gamma(s)} \int_0^{t_2} (t_2 - \tau)^{s-1} f(\tau, u(\tau)) \, d\tau \right| \\ &\quad = \left| \frac{1}{\Gamma(s)} \int_0^{t_1} (t_1 - \tau)^{s-1} f(\tau, u(\tau)) \, d\tau \right. \\ &\qquad - \frac{1}{\Gamma(s)} \int_0^{t_1} (t_2 - \tau)^{s-1} f(\tau, u(\tau)) \, d\tau \\ &\qquad \left. - \frac{1}{\Gamma(s)} \int_{t_1}^{t_2} (t_2 - \tau)^{s-1} f(\tau, u(\tau)) \, d\tau \right| \\ &\quad \le \frac{1}{\Gamma(s)} \int_0^{t_1} \left| (t_1 - \tau)^{s-1} - (t_2 - \tau)^{s-1} \right| \left| f(\tau, u(\tau)) \right| d\tau \\ &\qquad + \frac{1}{\Gamma(s)} \int_{t_1}^{t_2} (t_2 - \tau)^{s-1} \left| f(\tau, u(\tau)) \right| d\tau \end{aligned}$$

$$
\begin{aligned}
&\le \frac{L}{\Gamma(s)} \int_0^{t_1} \left((t_1 - \tau)^{s-1} - (t_2 - \tau)^{s-1} \right) d\tau \\
&\quad + \frac{L}{\Gamma(s)} \int_{t_1}^{t_2} (t_2 - \tau)^{s-1} \, d\tau \\
&= \frac{L}{\Gamma(s)} \left(\int_0^{t_1} (t_1 - \tau)^{s-1} \, d\tau - \int_0^{t_1} (t_2 - \tau)^{s-1} \, d\tau \right. \\
&\qquad\qquad\qquad\qquad \left. + \int_{t_1}^{t_2} (t_2 - \tau)^{s-1} \, d\tau \right) \\
&= \frac{L}{\Gamma(1+s)} \left(t_1^s + (t_2 - t_1)^s - t_2^s + (t_2 - t_1)^s \right) \\
&< \frac{2L}{\Gamma(1+s)} (t_2 - t_1)^s + \frac{L}{\Gamma(1+s)} (t_2^s - t_2^s) \\
&= \frac{2L}{\Gamma(1+s)} (t_2 - t_1)^s \\
&< \frac{2L}{\Gamma(1+s)} \delta^s \\
&= \varepsilon.
\end{aligned}
$$

Therefore, TM is equicontinuous. The Arzela–Ascoli Theorem implies that $\overline{TM}$ is compact. ∎

We introduce the following definition of a lower and upper solution for Eq. (1) (operator T).

DEFINITION. The function $v \in X$ is called a lower solution for Eq. (1) (operator T) if

$$D^s v(t) \le f(t, v(t)) (v(t) \le Tv(t)), \qquad 0 < t < 1.$$

Similarly, the function $w \in X$ is called an upper solution for Eq. (1) (operator T) if

$$D^s w(t) \ge f(t, w(t)) (w(t) \ge Tw(t)), \qquad 0 < t < 1.$$

If the strict inequalities hold, $v(t), w(t)$ are called strict lower and upper solutions.

Lower and upper solution can be defined in general semi-order Banach space see for example [7]. We need the following abstract theorem.

THEOREM A [5]. *Let D be a subset of the cone K of semi-order Banach space E, F: $D \to E$ be nondecreasing. If there exist $x_0, y_0 \in D$ such that $x_0 \leq y_0$, $\langle x_0, y_0 \rangle \subset D$ and x_0, y_0 are a lower and upper solution of equation $x - F(x) = 0$, then the equation $x - F(x) = 0$ has maximum solution and minimum solution x^*, y^* in $\langle x_0, y_0 \rangle$, such that $x^* \leq y^*$, when one of the following conditions holds*

(i) *K is normal and F is compact continuous*;

(ii) *K is regular and F is continuous*;

(iii) *E is reflexive, K is normal, and F is continuous or weak continuous*.

Now, we give the main results of this section.

THEOREM 2.1. *Assume*

(H_1) *f: $[0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $f(t, \cdot)$ is nondecreasing for each $t \in [0,1]$, and there exists a positive constant a, such that $f(t, \cdot)$ is strictly increasing on $[0, a]$ for each $t \in [0,1]$*;

(H_2) *v_0, w_0 are a lower and upper solution of Eq.* (1), *satisfying $v_0(t) \leq w_0(t)$, $0 \leq t \leq 1$.*

Then Eq. (1) has a positive solution.

Proof. We only need to consider the fixed point for operator T.

By Lemma 2.1, we have T: $K \to K$ is completely continuous, and obviously, v_0, w_0 are a lower and upper solution of T by the definition of T. By (H_1), $u_1, u_2 \in K$, $u_1 \leq u_2$, we have

$$\begin{aligned} Tu_1(t) &= \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, u_1(\tau))\, d\tau \\ &\leq \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, u_2(\tau))\, d\tau \\ &= Tu_2(t) \end{aligned}$$

Hence T is an increasing operator.

Obviously, we have $Tv_0 \geq v_0$, $Tw_0 \leq w_0$ by the definition of lower and upper solution of T, hence T: $\langle v_0, w_0 \rangle \to \langle v_0, w_0 \rangle$ is a compact continuous operator.

Since K is a normal cone, Theorem A implies that T has a fixed point $u \in \langle v_0, w_0 \rangle$. This completes the proof. ∎

Using the above theorem, we have the following theorem.

THEOREM 2.2. *Assume condition* (H_1) *in Theorem* 2.1 *holds*;

(H_2) $0 < \lim_{u \to +\infty} f(t, u(t)) < +\infty$ *for each* $t \in [0,1]$.

Then Eq. (1) has a positive solution.

Proof. By (H_2), there exist positive constants N, R, such that $f \leq N$, $\forall\ u \geq R$. Let $C = \max_{0 \leq t \leq 1,\, 0 \leq u \leq R} f(t,u)$. Then we have $f \leq N + C$, $\forall\ u \geq 0$.

Now, we consider the equation

$$D^s w(t) = N + C, \qquad 0 < s < 1, 0 < t < 1.$$

Obviously, the above equation has a solution

$$\begin{aligned} w(t) &= I^s(N + C) \\ &= \frac{1}{\Gamma(s)} \int_0^1 (t - \tau)^{s-1} (N + C)\, d\tau \\ &= \frac{N + C}{\Gamma(1 + s)} t^s \end{aligned}$$

and $w(t) = I^s(N + C) \geq I^s f(t, w(t))$, i.e., $w(t)$ is an upper solution of Eq. (1). Obviously, $v(t) \equiv 0$ is a lower solution of Eq. (1) and $0 \leq w(t)$, $0 \leq t \leq 1$.

Combining condition (H_1) and Theorem 2.1, we complete the proof. ∎

In order to get another result for a positive solution of Eq. (1), next, we consider the existence of a positive solution for the equation

$$D^s u(t) = Mu(t) + c, \qquad 0 < s < 1, 0 < t < 1,\ M, c > 0. \quad (2.2)$$

PROPOSITION. *Equation* (2.2) *has at least a positive solution.*

Proof. By [4, Proposition 2.4], (2.2) is equivalent to the integral equation

$$\begin{aligned} u(t) &= I^s(Mu(t) + c) \\ &= \frac{1}{\Gamma(s)} \int_0^t (t - \tau)^{s-1} (Mu(\tau) + c)\, d\tau. \end{aligned}$$

Let A: $K \to K$ be the operator defined as

$$Au(t) = \frac{1}{\Gamma(s)} \int_0^t (t - \tau)^{s-1} (Mu(\tau) + c)\, d\tau.$$

A is completely continuous by Lemma 2.1.

Let $B_R = \{u(t) \in (C[0,1], [0, +\infty)) \mid \|u - \frac{c}{\Gamma(1+s)} t^s\| \leq R\}$ be a convex, bounded, and closed subset of the Banach space $C[0, \delta]$, where $0 < \delta < 1$, $R > 0$.

We have, $\forall u \in B_R$

$$\begin{aligned}\left| Au(t) - \frac{c}{\Gamma(1+s)} t^s \right| &= \left| \frac{M}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} u(\tau)\, d\tau \right| \\ &\le \frac{M}{\Gamma(1+s)} \|u\| t^s.\end{aligned}$$

Since $\|u\| \le \dfrac{c}{\Gamma(1+s)} \delta^s + R \le \dfrac{c}{\Gamma(1+s)} + R$, hence

$$\begin{aligned}\left\| Au - \frac{c}{\Gamma(1+s)} t^s \right\| &\le \frac{M}{\Gamma(1+s)} \left(\frac{c}{\Gamma(1+s)} + R \right) \delta^s \\ &\le \tfrac{1}{2} R + \tfrac{1}{2} R = R,\end{aligned}$$

as long as let $\delta < \min\{(\Gamma(1+s)/2M)^{1/s}, (R(\Gamma(1+s))^2/2Mc)^{1/s}, 1\}$.

So we have $A(B_R) \subseteq B_R$.

The Schauder fixed point theorem assures that operator A has at least one fixed point and then Eq. (2.2) has at least one positive solution. ∎

We have the following existence theorem by the above proposition and Theorem 2.1.

THEOREM 2.3. *Assume condition* (H_1) *in Theorem* 2.1 *holds*;

(H_2) *Let* $0 \le \lim\limits_{u \to +\infty} \max\limits_{0 \le t \le 1} \dfrac{f(t,u)}{u} < +\infty$.

Then Eq. (1) has one positive solution.

3. A UNIQUE RESULT

In this section, we give the unique existence of a positive solution for Eq. (1) by using the following fixed theorem.

THEOREM B [6]. *Let P be a normal cone in Banach space E, B: $P \to E$ be a convex decreasing operator, and there exist a positive constant ε and natural number m, such that $\theta \le B\theta$, $\varepsilon B\theta \le B^2\theta$, $\frac{1}{2}B\theta \le B^{2m}\theta$. Then operator B has a unique fixed point x^* in P, and for arbitrary initial $x_0 \in [\theta, B\theta]$ given and iterative sequence $x_n = Bx_{n-1}$, $n = 1, 2, \ldots,$ there is $x_n \to x^*$.*

THEOREM 3.1. *Assume*

(H_1) *$f(t,\cdot)$ is decreasing convex for each $t \in [0,1]$;*

(H_2) *Let $0 < f(t,0) < \Gamma(1+s)$ $\forall$ $t \in [0,1]$;*

(H_3) *$f(t,1) \geq cf(t,0)$, $c \geq \frac{1}{2}$, $\forall$ $t \in [0,1]$.*

Then Eq. (1) has an unique positive solution.

Proof. T is a decreasing operator by condition (H_1), since for each $u, v \in K$, $0 \leq a \leq 1$

$$\begin{aligned}
&T(au + (1-a)v)(t) \\
&\quad = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, au(\tau) + (1-a)v(\tau))\, d\tau \\
&\quad \leq \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} (af(\tau, u(\tau)) + (1-a)f(\tau, v(\tau)))\, d\tau \\
&\quad = \frac{a}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, u(\tau))\, d\tau \\
&\qquad + \frac{1-a}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, v(\tau))\, d\tau \\
&\quad = aTu(t) + (1-a)Tv(t).
\end{aligned}$$

Hence T is a convex operator.
By (H_2) and (H_3), we have

$$T\theta(t) = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, 0)\, d\tau \geq \theta$$

$$\begin{aligned}
T\theta(t) &= \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, 0)\, d\tau \\
&< \frac{\Gamma(1+s)}{\Gamma(s)} \int_0^t (t-\tau)^{s-1}\, d\tau \\
&= \frac{\Gamma(1+s)}{\Gamma(1+s)} t^s \qquad \forall\ t \in [0,1]
\end{aligned}$$

$$\begin{aligned} T^2\theta(t) = T(T\theta(t)) &= \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, T\theta(\tau))\, d\tau \\ &\geq \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, 1)\, d\tau \\ &\geq \frac{c}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau, 0)\, d\tau \\ &= cT\theta(t). \end{aligned}$$

Hence $\theta \leq cT\theta \leq T^2\theta \leq T\theta$.

Theorem B implies that T has an unique fixed point; therefore, Eq. (1) has an unique positive solution. ∎

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